Concordia University February 9, 2007

**Engineering Differential Equations**
**Section J**
**Exam I (B)**

**ANSWER KEY**

(1) (10 points) For each of the following first-order differential equations, state whether it is separable, linear, homogeneous, or none of the above. You do not need to explain your answers.

Each equation may correspond to more than one listed type (in which case you must list **all** types) or none of those.

(a) $\dfrac{dy}{dx} = (x-1)(y-1)$ This DE is both linear and separable.

(b) $y' = y\sqrt{1 - \dfrac{y}{x}}$ This DE is none of the types listed above.

(c) $\dfrac{dy}{dx} = y\sqrt{1-x^2}$ This DE is both linear and separable.

(d) $x + 3y - xy' = 0$ This DE is both linear and homogeneous.

(e) $x\,\dfrac{dy}{dx} = \sqrt{x^2 - y^2}$ This DE is homogeneous.

(2) (10 points) Find the general solution of the first-order linear equation

$$\frac{dy}{dx} + 2y = e^{2x}.$$

**Solution:** We have the equation in its standard form

$$y' + 2y = e^{2x}, \quad \text{where} \quad y = y(x),$$

hence an integrating factor is $\mu(x) = e^{2x}$. Multiplying both sides of the equation by $\mu$, we obtain

$$e^{2x}y' + 2e^{2x}y = e^{4x}.$$

Equivalently

$$\left(e^{2x}y\right)' = e^{4x},$$

this integrating both sides of the equation with respect to $x$, we obtain

$$e^{2x}y = \frac{1}{4}\,e^{4x} + C, \quad C = \text{arbitrary constant.}$$

From here, the general solution of the given linear first-order differential equation is, in explicit form,

$$y(x) = \frac{1}{4}\,e^{2x} + C\,e^{-2x}, \quad C = \text{arbitrary constant.}$$

□

(3) Let

$$\left(\sqrt{x} + \frac{y}{x}\right) dx + (y^2 + \ln(2x))\, dy = 0, \quad x > 0.$$

(a) (5 points) Check that this is an exact differential equation.

(b) (5 points) Solve the equation leaving the general solution in implicit form.

**Solution:** (a) Let $M(x,y) = \sqrt{x} + \frac{y}{x}$ and let $N(x,y) = y^2 + \ln(2x)$. We note that

$$\frac{\partial M}{\partial y}(x,y) = \frac{1}{x} = \frac{\partial N}{\partial x}(x,y), \quad \text{for all } \ x > 0, y \in \mathbb{R}.$$

This proves that the given DE is exact.

(b) Consider $f(x,y)$ such that

$$\frac{\partial f}{\partial x}(x,y) = \sqrt{x} + \frac{y}{x}$$
$$\frac{\partial f}{\partial y}(x,y) = y^2 + \ln(2x).$$

Integrating the first equation with respect to $x$, we obtain

$$f(x,y) = \frac{2}{3}\sqrt{x^3} + y\ln(x) + C(y).$$

To find $C(y)$, we evaluate

$$\frac{\partial f}{\partial y}(x,y) = \ln(x) + C'(y) = y^2 + \ln(2x) = y^2 + \ln(2) + \ln(x),$$

hence $C(y)$ satisfies the ODE: $C'(y) = y^2 + \ln(2)$. Therefore

$$C(y) = \frac{y^3}{3} + y\ln(2) + c,$$

where $c$ is an arbitrary constant.

Finally we have that $f(x,y) = \frac{2}{3}\sqrt{x^3} + y\ln(x) + \frac{y^3}{3} + y\ln(2) + c$, and we conclude that the general solution of the given exact equation is (in implicit form)

$$\frac{2}{3}\sqrt{x^3} + y\ln(x) + \frac{y^3}{3} + y\ln(2) + c = 0, \quad \text{where } \ c \ \text{ is an arbitrary constant.}$$

□

(4) Suppose that $P(t)$ (with $t$ in months), the fish population in a lake contaminated by chemicals, satisfies the differential equation

$$\frac{dP}{dt} = -kP, \quad k > 0.$$

(a) (5 points) Find the general solution of this differential equation.

(b) (5 points) If today there are 400 fish in the pond, and we know that in a month there will be 300 fish in the pond, how many will be there after 1 year? (Give the exact value calculated with this model, do not approximate.)

**Solution:** (a) This ODE is a simple separable differential equation. Its general solution is $P(t) = ce^{-kt}$, where $c$ is an arbitrary constant, or in, an equivalent form, $P(t) = P_0 e^{-kt}$, where $P_0 = P(0)$. (To see this rewrite the equation in the form $\frac{1}{P}\frac{dP}{dt} = -k$ and integrate both sides. Afterwards, apply the exponential function to both sides of the resulting equality.)

(b) Based on the form $P(t) = P_0 e^{-kt}$, we immediately get that $P(t) = 400e^{-kt}$. We need to use the second condition to find the value of the constant $k$. We also know that

$$P(1) = 400e^{-k} = 300,$$

thus $e^{-k} = 3/4$ and $k = -\ln(3/4)$.

We conclude that

$$P(t) = 400e^{t\ln(3/4)} = 400\left(\frac{3}{4}\right)^t.$$

Consequently $P(12) = 400\left(\frac{3}{4}\right)^{12}$ which is the exact answer. (Of course, $P(12) \approx 12$ fish. However, note that the question was asking for the exact value of the solution to the ODE at time $t = 12$.)

□