Concordia University February 13, 2009

Applied Differential Equations
Section J
Exam I (B)

ANSWER KEY

(1) (10 points) The ODE

$$x^2 \frac{dy}{dx} = y(1-x)$$

is separable. It can be written in the form

$$\frac{dy}{y} = \frac{1-x}{x^2}\, dx,$$

which, by integration leads to the solution in implicit form

$$\ln \mid y \mid = -\frac{1}{x} - \ln \mid x \mid + C,$$

where C is an arbitrary constant.

□

(2) (10 points) As the equation $y\,dx - (x+y)\,dy = 0$ is homogeneous, we proceed with the substitution $y = ux$ (thus $\frac{dy}{dx} = \frac{du}{dx}\,x + u$).

We obtain the following ODE for the dependent variable u:

$$x\,\frac{du}{dx} = \frac{u}{u+1} - u.$$

Furthermore, this is a separable ODE, namely $x\,\frac{du}{dx} = -\frac{u^2}{u+1}$. We re-write it as $\frac{u+1}{u^2}\,du = -\frac{1}{x}\,dx$ and we integrate both sides to obtain:

$$-\frac{1}{u} + \ln|u| = -\ln|x| + C, \quad C \text{ arbitrary constant.}$$

Recall that $u = \frac{y}{x}$ which leads to the implicit solution of the original ODE in the form

$$-\frac{x}{y} + \ln|y/x| = -\ln|x| + C, \quad C \text{ arbitrary constant.}$$

Note: One can use the identity $\ln(a/b) = \ln a - \ln b$ to simplify the previous form of the solution to $-\frac{x}{y} + \ln|y| = C, \quad C$ arbitrary constant.

□

(3) Given

$$y(y+\sin x)\,dx + \left(2xy - \cos x + \frac{1}{\sqrt{y+3}}\right) dy = 0,$$

we consider $f(x,y)$ such that

$$\frac{\partial f}{\partial x}(x,y) = y(y+\sin x)$$
$$\frac{\partial f}{\partial y}(x,y) = 2xy - \cos x + \frac{1}{\sqrt{y+3}}.$$

Integrating the first equation with respect to x, we obtain

$$f(x,y) = y^2x - y\cos x + c(y).$$

To find $c(y)$, we evaluate

$$\frac{\partial f}{\partial y}(x,y) = 2yx - \cos x + c'(y) = 2xy - \cos x + \frac{1}{\sqrt{y+3}},$$

hence $c(y)$ satisfies the ODE: $c'(y) = \frac{1}{\sqrt{y+3}}$. Therefore $c(y) = 2\sqrt{y+3} + C$, where C is an arbitrary constant.

Finally we have that $f(x,y) = y^2x - y\cos x + 2\sqrt{y+3} + C$, and we conclude that the general solution of the given exact equation is (in implicit form)

$$y^2x - y\cos x + 2\sqrt{y+3} = c, \quad \text{where } c \text{ is an arbitrary constant.}$$

Now, $y(0) = 1$ implies $c = 3$, hence the solution of the IVP

$$y^2x - y\cos x + 2\sqrt{y+3} = 3.$$

□

(4) Denote by $A(t)$ the number of pounds of salt in the tank at time t. Then we must solve the IVP

$$\frac{dA}{dt} = 10 - \frac{4A}{150+t}, \quad A(0) = 0.$$

The ODE is linear with an integrating factor $\mu(t) = \exp\left(\int \frac{4}{150+t}\,dt\right) = \exp(4\ln(150+t)) = (150+t)^4$.

So, $(150+t)^4A'(t) + 4A(150+t)^3 = 10(150+t)^4$, hence $(150+t)^4A(t) = \frac{10}{5}(150+t)^5 + C$ where C will be determined from $A(0) = 0$ to be equal to $-2\cdot 150^5$.

Finally, we have $A(t) = 2(150+t) - 2\cdot 150^5\,(150+t)^{-4}$ and $A(30) = 360 - 300\left(\frac{150}{180}\right)^4 \approx 215.32$ lb.

□